of complete ionization; but in closing it may be pointed out that it accounts for the remarkable facts that so many very dissimilar chemical ubstances (for example, hydrochloric acid and potassium chloride) seem to be equally ionized, and that a volatile substance like hydrochloric acid does not have an appreciable vapor-pressure even in 1-normal solution where 15% of it must be assumed to be in the un-ionized state, if the conductance-ratio is taken as a measure of ionization. It may also be mentioned that it avoids the improbable conclusions as to the abnormal activity of the un-ionized molecules to which solubility-effects, interpreted under the older assumptions, lead.[12]

[1] Lewis, *Proc. Amer. Acad.*, **43,** 1907 (259–293); *Zs. physik. Chem.*, **61,** 1908 (129–165).

[2] MacInnes and Parker, *J. Amer. Chem. Soc.*, **37,** 1915 (1445–1461).

[3] Ellis, *Ibid.*, **38,** 1916 (737–762); Noyes and Ellis, *Ibid.*, **39,** 1917 (2532–2544).

[4] Jahn, *Zs. physik. Chem.*, **33,** 1900 (559–576).

[5] Harned, *J. Amer. Chem. Soc.*, **38,** 1916 (1989).

[6] Linhart, *Ibid.*, **41,** 1919 (1175–1180).

[7] Lewis, *Ibid.*, **34,** 1912 (1635); Bates, *Ibid.*, **37,** 1915 (1421–1445).

[8] Noyes and Falk, *Ibid.*, **33,** 1911 (1454).

[9] MacInnes, *Ibid.*, **41,** 1919 (1086).

[10] Noyes, "The Physical Properties of Aqueous Salt Solutions in Relation to the Ionic Theory," *Congress of Arts and Sciences, St. Louis Exposition*, **4,** 1904 (317); *Science*, **20,** 1904 (582); abstract, *Zs. physik. Chem.*, **52** (635); also Noyes, *J. Amer. Chem. Soc.*, **30,** 1908 (335–353).

[11] Milner, *Phil. Mag.*, **35,** 1918 (214, 354); Ghosh, *Trans. Chem. Soc. (London)*, **113,** 1918 (449, 627); Bjerrum, *Zs. Elektrochem.*, **24,** 1918 (321).

[12] Bray, *J. Amer. Chem. Soc.*, **33,** 1911 (1673–1686).

THE COMMUTATIVITY OF ONE-PARAMETER TRANSFORMATIONS IN REAL VARIABLES

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Communicated by E. H. Moore, November 10, 1919

If $X = \Sigma\xi_i(x_1 \ldots x_n)\dfrac{\partial}{\partial x_i}$ and $Y = \Sigma\eta_i(x_1 \ldots x_n)\dfrac{\partial}{\partial x_i}$ be the differential symbols of two one-parameter transformations, the known condition for the commutativity of the finite transformations is the identical vanishing of the commutator $(XY) = XY - YX$. The proof given by Lie-Engel, applying to the case of analytic functions, depends on expansions in powers of the canonical parameters. The computation of (XY) according to the original definition involves the use of the second partial derivatives of the coefficients ξ, η, but the final form, containing only first derivatives, is

$$(XY) = \sum_i (X\eta_i - Y\xi_i)\frac{\partial}{\partial x_i}. \tag{1}$$

The proof here sketched, in which the commutator is to be considered defined directly by (1), shows that for the case of real variables it is sufficient to assume the existence of continuous, first partial derivatives only of the ξ's and η's. The basis of the proof is formed by familiar theorems on the existence and uniqueness of the solutions of differential equations and their differentiability with respect to parameters. A minimum range of variation for which the results can be known to apply could be specified by the use of inequalities such as naturally occur in connection with those theorems.

With the canonical parameters denoted by α and β, and the corresponding finite transformations by X_α and Y_β, let the point $(x_{10}, \ldots, x_{n0})$ be transformed first by Y_β then by X_α, defining thus the functions $x_i(\alpha, \beta)$. These are then determined by the differential equations

$$D_\alpha x_i(\alpha,\beta) = \xi_i[x(\alpha,\beta)], \tag{2}$$

in which the values initial for $\alpha = 0$ are determined by

$$D_\beta x_i(0,\beta) = \eta_i[x(0,\beta)], \tag{3}$$

where x_i (0, 0) is to be x_{io}. The functions x_i (α, β) then exist for a certain range of variation of α and β; and with respect to the latter they have derivatives satisfying the equations

$$D_\alpha D_\beta x_i(\alpha,\beta) = \sum_j \xi_{ij}[x(\alpha,\beta)]D_\beta x_j(\alpha,\beta), \tag{4}$$

in which ξ_{ij} means $\partial\xi_i/\partial x_j$. Now $D_\alpha\eta_i[x(\alpha,\beta)]$ exists and is given by

$$D_\alpha\eta_i[x(\alpha,\beta)] = \sum_j \eta_{ij}[x(\alpha,\beta)]\xi_j[x(\alpha,\beta)], \tag{5}$$

which by virtue of the assumed identical vanishing of (XY) is equivalent to

$$D_\alpha\eta_i[x(\alpha,\beta)] = \sum_j \xi_{ij}[x(\alpha,\beta)]\eta_j[x(\alpha,\beta)]. \tag{6}$$

Comparison of (4) and (6) shows that the $D_\beta x_i(\alpha,\beta)$ and the $\eta_i[x(\alpha,\beta)]$ satisfy the same system of linear differential equations, and according to (3) they agree for $\alpha = 0$. Hence, for every α,β,

$$D_\beta x_i(\alpha,\beta) = \eta_i[x(\alpha,\beta)]. \tag{7}$$

Thus the functions $x_i(\alpha,\beta)$ are the same as would be obtained by the opposite sequence, X_α followed by Y_β, showing the condition in question to be sufficient.

A sense in which the vanishing of (XY) is also necessary appears likewise in connection with the differential equations. If for both orders of

transformation the $x_i(\alpha,\beta)$ be the same, then corresponding to those orders is to be inferred the existence of the derivatives

$$D_\alpha D_\beta x_i = \sum_j \xi_{ij} D_\beta x_j = \sum_j \eta_j \xi_{ij} = Y\xi_i,$$

$$D_\beta D_\alpha x_i = \sum_j \eta_{ij} D_\alpha x_j = \sum_j \xi_j \eta_{ij} = X\eta_i.$$

If these are to agree the commutator must vanish at all points actually occurring in the transformation; a similar limitation in the meaning of "identical" may be understood in the proof of sufficiency.

THE INTENSITIES OF X-RAYS OF THE L SERIES
II. THE CRITICAL POTENTIALS OF THE PLATINUM LINES

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Communicated by Edwin B. Wilson. Read before Academy, November 10, 1919

Introduction.—This work is a continuation of that of Webster and Clark, reported in these PROCEEDINGS[1] in 1917. Part of the present work was done at Harvard University with the apparatus described in the earlier paper, and part with my new apparatus at the Massachusetts Institute of Technology. The object in view is the investigation of the laws relating intensity to potential, for the L-series lines, for the purpose of comparison with current theories of X-ray spectra, and the present paper deals with the determination of the critical potentials of the platinum lines of medium intensity, the stronger ones having been reported in the previous paper and the fainter ones, observed only in tungsten by Dershem[2] and Overn,[3] being so faint as to require a much more prolonged study.

Apparatus.—The work at Harvard showed that with the slit widths needed for accurate intensity measurements it would be difficult to work with certainty on any lines but the strong ones, when the voltage was near the critical value. As the previous work of the author on the rhodium K series[4] had shown that photography gave good results in such work it was decided to use it here. In this case, where the lines are many and scattered and faint, the best instrument seemed to be the bent mica spectrograph of de Broglie and Lindemann[5] with which they have obtained excellent spectrograms. These show spectra of six orders, called first to sixth inclusive, though I think the last one, from its angle, must really be the seventh rather than the sixth, which must be very faint. The grating space is about 10 Ångstroms and the third and fifth orders are the strongest.